

# Some New Inequalities of Dirichlet Eigenvalues for Laplace Operator with any Order

Na Huang<sup>\*</sup>, Pengcheng Niu<sup>†</sup>

**Abstract.** In this paper, we establish several inequalities of Dirichlet eigenvalues for Laplace operator  $\Delta$  with any order on  $n$ -dimensional Euclidean space. These inequalities are more general than known Yang's inequalities and contain new consequences. To obtain them, we borrow the approach of Illias and Makhoul, and use a generalized Chebyshev's inequality.

**Keywords.** Laplace operator; Dirichlet eigenvalue; inequality

## 1 Introduction

The following Dirichlet problem

$$\begin{cases} (-\Delta)^l u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

has been extensively considered, where  $\Delta$  is the Laplacian:  $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ ,  $l$  is any positive integer,  $\Omega$  is a bounded domain in the Euclidean space  $R^n$ ,  $\nu$  is the outward unit normal on  $\partial\Omega$ .

When  $l = 1$ , Payne, Pólya and Weinberger in [9] showed the following inequality (the

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<sup>\*</sup>Na Huang: Department of Applied Mathematics; Key Laboratory of Space Applied Physics and Chemistry, Ministry of Education, Northwestern Polytechnical University, Xi'an, Shaanxi, 710129, China. e-mail: huangna7@126.com

<sup>†</sup>Pengcheng Niu: Corresponding author, Department of Applied Mathematics; Key Laboratory of Space Applied Physics and Chemistry, Ministry of Education, Northwestern Polytechnical University, Xi'an, Shaanxi, 710129, China. e-mail: pengchengniu@nwpu.edu.cn

PPW inequality)

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{r=1}^k \lambda_r.$$

The inequality (the HP inequality)

$$\sum_{r=1}^k \frac{\lambda_r}{\lambda_{k+1} - \lambda_r} \geq \frac{nk}{4}$$

was due to Hile and Protter in [4]. Yang in [10] proved some important eigenvalue estimates, which are Yang's first inequality

$$\sum_{r=1}^k (\lambda_{k+1} - \lambda_r)^2 \leq \frac{4}{n} \sum_{r=1}^k (\lambda_{k+1} - \lambda_r) \lambda_r$$

and Yang's second inequality

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{r=1}^k \lambda_r.$$

When  $l = 2$ , the estimate

$$\lambda_{k+1} - \lambda_k \leq \frac{8(n+2)}{n^2 k} \sum_{r=1}^k \lambda_r$$

was derived by Payne, Plya and Weinberger in [9]. Chen and Qian in [1] and Hook in [5] proved respectively

$$\frac{n^2 k^2}{8(n+2)} \leq \sum_{i=1}^k \frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^k \lambda_i^{\frac{1}{2}}.$$

The following inequality

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \leq \left(\frac{8(n+2)}{n^2}\right)^{\frac{1}{2}} \sum_{i=1}^k (\lambda_i (\lambda_{k+1} - \lambda_i))^{\frac{1}{2}}$$

was gotten by Cheng and Yang [3].

For any positive integer  $l$ , Chen and Qian [1] and Hook [5] independently obtained

$$\frac{n^2 k^2}{4l(n+2l-2)} \leq \sum_{i=1}^k \frac{\lambda_i^{\frac{1}{l}}}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^k \lambda_i^{\frac{l-1}{l}}, k = 1, 2, \dots.$$

The inequality

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4l(n+2l-2)}{n^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i$$

was concluded by Cheng, Ichikawa and Mametsuka in [2]. Ilias and Makhoul in [7] exhibited a new abstract formula relating eigenvalues of a self-adjoint operator and deduced Yang type inequality for Dirichlet eigenvalues of sub-Laplacian with any order on the Heisenberg group, in the light of Chebyshev's inequality.

In this paper, we will give several new estimates of Dirichlet eigenvalues to (1.1) by combining the approach of Ilias and Makhoul in [8] and using a generalized Chebyshev's inequality in [6]. For convenience, we denote  $L = -\Delta$  and assume always that  $\lambda_{i+1} > \lambda_i, i = 1, 2, \dots$ , in the sequel. The main results of this paper are the following Theorem 1.1 and its corollaries.

**Theorem 1.1** Let  $\{\lambda_i\}$  be the eigenvalues of (1.1), then

$$\begin{aligned} & \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \\ & \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{l-1}{l}} \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{l}} \right]^{\frac{1}{2}}, \end{aligned} \quad (1.2)$$

where  $\alpha \in \mathbb{R}$  and  $\beta \geq 0$  such that  $\alpha^2 \leq 2\beta$ .

**Remark 1.1** Inequality (1.2) is the generalization of Yang's inequality. Some consequences are easily deduced from (1.2) and new inequalities are listed now.

(1) Let  $2\alpha - \beta - 1 = 0$ , then

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-1} \lambda_i^{\frac{l-1}{l}} \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right]^{\frac{1}{2}},$$

where  $\alpha \in [2 - \sqrt{2}, 2 + \sqrt{2}]$ .

(2) When  $\alpha = \beta = 1$ , it follows

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i^{\frac{l-1}{l}} \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right]^{\frac{1}{2}}.$$

(3) When  $\alpha = \frac{1}{2}$ , and  $\beta \geq \frac{1}{8}$ , we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{l-1}{l}} \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{-\beta} \lambda_i^{\frac{1}{l}} \right]^{\frac{1}{2}}.$$

(4) When  $\alpha = -1$ , and  $\beta \geq \frac{1}{2}$ , then

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{-1} \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{l-1}{l}} \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{-\beta-3} \lambda_i^{\frac{1}{l}} \right]^{\frac{1}{2}}.$$

We note that the forms in (3) and (4) are never seen previously.

**Corollary 1.1** We have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \lambda_i^{\frac{l-1}{l}} \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i^{\frac{1}{l}} \right]^{\frac{1}{2}}. \quad (1.3)$$

**Corollary 1.2** It holds

$$\lambda_{k+1} - \lambda_k \leq \frac{4l(n+2l-2)}{n^2 k^2} \left( \sum_{i=1}^k \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right). \quad (1.4)$$

**Corollary 1.3** We have

$$\begin{aligned} & \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \\ & \leq \frac{2\sqrt{l(n+2l-2)}}{n} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \right]^{\frac{1}{2}} \left[ \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i \right]^{\frac{1}{2}}, \end{aligned} \quad (1.5)$$

where  $\alpha \in R, \beta \geq 0$  and  $\alpha^2 \leq 2\beta$ .

**Corollary 1.4** Yang type first inequality for (1.1) holds:

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4l(n+2l-2)}{n^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i. \quad (1.6)$$

**Corollary 1.5** We have the Payne-Pólya-Weinberger Type inequality

$$\lambda_{k+1} - \lambda_k \leq \frac{4l(n+2l-2)}{n^2 k} \sum_{i=1}^k \lambda_i. \quad (1.7)$$

**Corollary 1.6** The Yang type second inequality holds:

$$\lambda_{k+1} \leq \left( 1 + \frac{4l(n+2l-2)}{n^2} \right) \frac{1}{k} \left( \sum_{i=1}^k \lambda_i \right). \quad (1.8)$$

This paper is arranged as follows. Section 2 is devoted to the description of known results and some elementary inequalities. The proofs of Theorem 1.1 and Corollaries 1.1-1.6 are given in Section 3.

## 2 Preliminaries

**Definition 2.1**(see [7]) A couple  $(f, g)$  of functions  $f$  and  $g$  on the interval  $(0, \lambda)$  ( $\lambda > 0$ ) is said to belong to  $\chi_\lambda$  provided that

- (1)  $f$  and  $g$  are positive;  
(2)  $f$  and  $g$  satisfy

$$\left(\frac{f(x) - f(y)}{x - y}\right)^2 + \left(\frac{(f(x))^2}{g(x)(\lambda - x)} + \frac{(f(y))^2}{g(y)(\lambda - y)}\right) \left(\frac{g(x) - g(y)}{x - y}\right) \leq 0,$$

for any  $x, y \in (0, \lambda)$ ,  $x \neq y$ .

**Lemma 2.1**(see [6]) Let  $(f, g) \in \chi_\lambda$ , then  $g$  must be nonincreasing; if  $f(x) = (\lambda - x)^\alpha$ ,  $g(x) = (\lambda - x)^\beta$ , then  $\alpha^2 \leq 2\beta$ .

**Definition 2.2**(see [7]) For any two operators  $A$  and  $B$ , their commutator  $[A, B]$  is defined by  $[A, B] = AB - BA$ .

**Lemma 2.2** For  $p = 1, 2, \dots, n$ , we have

$$L^l(x_p u_i) = x_p L^l u_i - 2l L^{l-1} \frac{\partial}{\partial x_p} u_i, \quad (2.1)$$

$$[L^l, x_p] u_i = -2l L^{l-1} \frac{\partial}{\partial x_p} u_i. \quad (2.2)$$

**Proof.** When  $l = 1$ , we have

$$\frac{\partial}{\partial x_j} (x_p u_i) = \left(\frac{\partial}{\partial x_j} x_p\right) u_i + x_p \left(\frac{\partial}{\partial x_j} u_i\right)$$

and

$$\frac{\partial^2}{\partial x_j^2} (x_p u_i) = 2 \left(\frac{\partial}{\partial x_j} x_p\right) \left(\frac{\partial}{\partial x_j} u_i\right) + x_p \left(\frac{\partial^2}{\partial x_j^2} u_i\right).$$

Hence

$$L(x_p u_i) = (-\Delta)(x_p u_i) = x_p L u_i - 2 \frac{\partial}{\partial x_p} u_i,$$

and (2.1) is proved.

Assuming (2.1) is true for  $l - 1$ , direct calculations show

$$L^{l-1}(x_p u_i) = x_p L^{l-1} u_i - 2(l-1) L^{l-2} \frac{\partial}{\partial x_p} u_i$$

and

$$\begin{aligned} L^l(x_p u_i) &= L(L^{l-1}(x_p u_i)) \\ &= L(x_p L^{l-1} u_i - 2(l-1) L^{l-2} \frac{\partial}{\partial x_p} u_i) \\ &= x_p L^l u_i - 2l L^{l-1} \frac{\partial}{\partial x_p} u_i. \end{aligned}$$

So (2.1) is valid for  $l$ .

Noting

$$[L^l, x_p] u_i = L^l(x_p u_i) - x_p L^l u_i = -2l L^{l-1} \frac{\partial}{\partial x_p} u_i,$$

it follows (2.2).

**Lemma 2.3**(see [7]) Let  $A: D \subset H \rightarrow H$  be a self-adjoint operator defined on a dense domain  $D$ , which is semibounded below and has a discrete spectrum  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ . Let  $\{T_p: D \rightarrow H\}_{p=1}^n$  be a collection of skew-symmetric operators and  $\{B_p: T_p(D) \rightarrow H\}_{p=1}^n$  a collection of symmetric operators, leaving  $D$  invariant. We denote by  $\{u_i\}_{i=1}^n$  a basis of orthonormal eigenvectors of  $A$ ,  $u_i$  corresponding to  $\lambda_i$  and let  $\lambda_{k+1} \geq \lambda_k$ ,  $k \geq 1$ . Then for any  $(f, g)$  in  $\chi_{\lambda_{k+1}}$ , it follows

$$\begin{aligned} & \left( \sum_{i=1}^k \sum_{p=1}^n f(\lambda_i) \langle [T_p, B_p] u_i, u_i \rangle \right)^2 \\ & \leq 4 \left( \sum_{i=1}^k \sum_{p=1}^n g(\lambda_i) \langle [A, B_p] u_i, B_p u_i \rangle \right) \left( \sum_{i=1}^k \sum_{p=1}^n \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)} \|T_p u_i\|^2 \right). \end{aligned} \quad (2.3)$$

**Lemma 2.4**(see [6]) For  $\gamma \geq 1$ ,  $s_i \geq 0$ ,  $i = 1, \dots, k$ , it follows

$$\left( \sum_{i=1}^k s_i \right)^\gamma \leq k^{\gamma-1} \sum_{i=1}^k s_i^\gamma.$$

**Lemma 2.5**(Chebyshev's inequality, [8]) If  $(a_k - a_j)(b_k - b_j) \leq 0$  for any nonnegative  $k, j$ , then

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{n} \left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right).$$

**Lemma 2.6**(generalized Chebyshev's inequality, see [6]) If  $A_1 \geq A_2 \geq \dots \geq A_k \geq 0$ ,  $0 \leq B_1 \leq B_2 \leq \dots \leq B_k$ ,  $0 \leq C_1 \leq C_2 \leq \dots \leq C_k$ , then it implies that for  $\alpha^2 \leq 2\beta$ ,

$$\sum_{i=1}^k A_i^\beta B_i \sum_{i=1}^k A_i^{2\alpha-\beta-1} C_i \leq \sum_{i=1}^k A_i^\beta \sum_{i=1}^k A_i^{2\alpha-\beta-1} B_i C_i. \quad (2.4)$$

By Lemma 2.6, we immediately have

**Corollary 2.1**(see [2]) If  $A_1 \geq A_2 \geq \dots \geq A_k \geq 0$ ,  $0 \leq B_1 \leq B_2 \leq \dots \leq B_k$ ,  $0 \leq C_1 \leq C_2 \leq \dots \leq C_k$ , then we have that for  $\alpha^2 \leq 2\beta$ ,

$$\sum_{i=1}^n A_i^2 B_i \sum_{i=1}^n A_i C_i \leq \sum_{i=1}^n A_i^2 \sum_{i=1}^n A_i B_i C_i.$$

**Lemma 2.7**(see [1]) Let  $\lambda_i$ ,  $i = 1, 2, \dots$ , be the eigenvalues of (1.1), and  $u_i$  the corresponding eigenfunctions, then

$$\int_{\Omega} u_i L^k u_i = \int_{\Omega} |\nabla^k u_i|^2 \leq \left( \int_{\Omega} u_i L^l u_i \right)^{\frac{k}{l}} = \lambda_i^{\frac{k}{l}}, k = 1, \dots, l-1,$$

where

$$\nabla^k \equiv \begin{cases} \Delta^{\frac{k}{2}}, & \text{if } k \text{ is even,} \\ \nabla \left( \Delta^{\frac{k-1}{2}} \right), & \text{if } k \text{ is odd.} \end{cases}$$

### 3 Proofs of results

**Proof of Theorem 1.1.** We apply (2.3) with  $A = L^l = (-\Delta)^l$ ,  $B_1 = x_1, \dots, B_n = x_n$ ,  $T_1 = \frac{\partial}{\partial x_1}, \dots, T_n = \frac{\partial}{\partial x_n}$ ,  $f(x) = (\lambda - x)^\alpha$ ,  $g(x) = (\lambda - x)^\beta$ , and obtain

$$\begin{aligned} & \left( \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\alpha \left\langle \left[ \frac{\partial}{\partial x_p}, x_p \right] u_i, u_i \right\rangle_{L^2} \right)^2 \\ & \leq 4 \left( \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\beta \langle [L^l, x_p] u_i, x_p u_i \rangle_{L^2} \right) \\ & \times \left( \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \left\| \frac{\partial}{\partial x_p} u_i \right\|_{L^2}^2 \right). \end{aligned} \quad (3.1)$$

Since

$$\left[ \frac{\partial}{\partial x_p}, x_p \right] u_i = \frac{\partial}{\partial x_p} (x_p u_i) - x_p \frac{\partial}{\partial x_p} u_i = u_i,$$

and

$$\left\langle \left[ \frac{\partial}{\partial x_p}, x_p \right] u_i, u_i \right\rangle_{L^2} = 1,$$

it arrives at

$$\left( \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\alpha \left\langle \left[ \frac{\partial}{\partial x_p}, x_p \right] u_i, u_i \right\rangle_{L^2} \right)^2 = \left( n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \right)^2. \quad (3.2)$$

By (2.1) and (2.2), it follows

$$\begin{aligned} & \langle [L^l, x_p] u_i, x_p u_i \rangle_{L^2} \\ & = -2l \int_{\Omega} x_p u_i L^{l-1} \frac{\partial}{\partial x_p} u_i = -2l \int_{\Omega} \frac{\partial}{\partial x_p} u_i L^{l-1} (x_p u_i) \\ & = -2l \int_{\Omega} \frac{\partial}{\partial x_p} u_i \left\{ x_p L^{l-1} u_i - 2(l-1) L^{l-2} \frac{\partial}{\partial x_p} u_i \right\} \\ & = 2l \int_{\Omega} x_p u_i L^{l-1} \frac{\partial}{\partial x_p} u_i + 2l \int_{\Omega} u_i L^{l-1} u_i - 4l(l-1) \int_{\Omega} u_i L^{l-2} \frac{\partial^2}{\partial x_p^2} u_i, \end{aligned}$$

hence

$$\langle [L^l, x_p] u_i, x_p u_i \rangle_{L^2} = l \int_{\Omega} u_i L^{l-1} u_i - 2l(l-1) \int_{\Omega} u_i L^{l-2} \frac{\partial^2}{\partial x_p^2} u_i.$$

We see from Lemma 2.7 that

$$\begin{aligned}
& \sum_{p=1}^n \langle [L^l, x_p] u_i, x_p u_i \rangle_{L^2} \\
&= l(2l+n-2) \int_{\Omega} u_i L^{l-1} u_i \\
&\leq l(2l+n-2) \left( \int_{\Omega} u_i L^l u_i \right)^{\frac{l-1}{l}} \\
&= l(2l+n-2) \lambda_i^{\frac{l-1}{l}}
\end{aligned}$$

and

$$\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{\beta} \langle [L^l, x_p] u_i, x_p u_i \rangle_{L^2} \leq l(2l+n-2) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{l-1}{l}}. \quad (3.3)$$

Since

$$\sum_{p=1}^n \left\| \frac{\partial}{\partial x_p} u_i \right\|_{L^2}^2 = \int_{\Omega} u_i L u_i \leq \left( \int_{\Omega} u_i L^l u_i \right)^{\frac{1}{l}} = \lambda_i^{\frac{1}{l}},$$

it yields

$$\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \left\| \frac{\partial}{\partial x_p} u_i \right\|_{L^2}^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{l}}. \quad (3.4)$$

Instituting (3.2), (3.3) and (3.4) into (3.1), we have

$$\left( n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \right)^2 \leq 4l(2l+n-2) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{l}} \right)$$

and Theorem 1.1 is proved.

**Proof of Corollary 1.1.** To obtain (1.3), it suffices to take  $\alpha = \beta = 2$  in (1.2).

**Proof of Corollary 1.2.** When  $1 \leq \alpha = \beta \leq 2$ , we have from (1.2) that

$$\left( n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \right)^2 \leq 4l(2l+n-2) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{l}} \right).$$

Applying Lemma 2.5 to  $\left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\beta} \lambda_i^{\frac{l-1}{l}} \right)$  and  $\left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{l}} \right)$ , it follows

$$\begin{aligned}
& \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \right)^2 \\
& \leq \frac{4l(2l+n-2)}{n^2 k^2} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\beta} \right) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right) \\
& = \frac{4l(2l+n-2)}{n^2 k^2} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \right) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right)
\end{aligned}$$



and then

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{4l(2l+n-2)}{n^2 k^2} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right),$$

so

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \left( (\lambda_{k+1} - \lambda_k) - \frac{4l(2l+n-2)}{n^2 k^2} \left( \sum_{i=1}^k \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k \lambda_i^{\frac{1}{l}} \right) \right) \leq 0.$$

Since  $\lambda_i \leq \lambda_k$  for all  $i \leq k$ , we have (1.4).

**Proof of Corollary 1.3.** We have from Theorem 1.1 that

$$\left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \right)^2 \leq \frac{4l(2l+n-2)}{n^2} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{l-1}{l}} \right) \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{l}} \right)$$

and show (1.5).

**Proof of Corollary 1.4.** Let us take  $\alpha = \beta = 2$  in (1.5) to obtain (1.6).

**Proof of Corollary 1.5.** When  $\alpha = \beta = 2$ , we know from Corollary 1.3 that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{4l(n+2l-2)}{n^2} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \right). \quad (3.5)$$

Using Lemmas 2.4 and 2.5, it implies

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \geq \frac{1}{k^{\alpha-1}} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^\alpha \geq \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha-1} (\lambda_{k+1} - \lambda_k)$$

and

$$\frac{4l(n+2l-2)}{n^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \leq \frac{4l(n+2l-2)}{n^2 k} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left( \sum_{i=1}^k \lambda_i \right),$$

then from (3.5) that

$$\left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha-1} (\lambda_{k+1} - \lambda_k) \leq \frac{4l(n+2l-2)}{n^2 k} \left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left( \sum_{i=1}^k \lambda_i \right).$$

Since

$$\left( \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha-1} \geq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1},$$

we have (1.7).

**Proof of Corollary 1.6.** When  $1 \leq \alpha = \beta \leq 2$ , we have from Corollary 1.3 that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{4l(n+2l-2)}{n^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i,$$

then

$$\lambda_{k+1} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} - \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \leq \frac{4l(n+2l-2)}{n^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i,$$

i.e.,

$$\lambda_{k+1} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \leq \left(1 + \frac{4l(n+2l-2)}{n^2}\right) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i.$$

By Lemma 2.5, it follows

$$\begin{aligned} & \left(1 + \frac{4l(n+2l-2)}{n^2}\right) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \\ & \leq \left(1 + \frac{4l(n+2l-2)}{n^2}\right) \frac{1}{k} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1}\right) \left(\sum_{i=1}^k \lambda_i\right) \end{aligned}$$

and

$$\left(\lambda_{k+1} - \left(1 + \frac{4l(n+2l-2)}{n^2}\right) \frac{1}{k} \left(\sum_{i=1}^k \lambda_i\right)\right) \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1}\right) \leq 0.$$

Since  $\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1}\right) \geq 0$ , it implies (1.8).

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